BASIC COUNTERFACTUAL MODEL (CHAP 1-3)

Attached to every individual:

y = observed outcome
A = treatment indicator (0 or 1)
L = covariates
y^1 = outcome if treated
y^0 = outcome if not treated

$\left.\right\}$ Counterfactuals or Potential outcomes

A causal effect exists if y^0 and y^1 are not identically distributed

Average causal effect (in population) $Ey^1 - Ey^0$
(or ratio, or odds ratio).

(C) CONSISTENCY $y = y^A$ i.e. $y(\omega) = y^{A(\omega)}(\omega)$

(CE) CONDITIONAL EXCHANGEABILITY $y^a \perp\!\!\!\perp A \mid L$, $\forall a$.

(P) POSITIVITY $P(A=a \mid L) > 0$ a.s., $\forall a$.

THM under (C)+(CE)+(P)

$$Ey^a = E_L\, E(y \mid L, A=a) \quad \text{(standardization)}$$

$$= E\left[\frac{y 1_{A=a}}{f(a \mid L)}\right] \quad \text{(IP weighting)}$$

$f(a \mid L) = P(A=a \mid L)$

proof $Ey^a = E_L E(y^a \mid L) \underset{(CE)+(P)}{=} E_L E(y^a \mid L, A=a) = E_L E(y^A \mid L, A=a)$
$\overset{(C)}{=} E_L E(y \mid L, A=a)$

$$= E_L\left[\frac{E(y 1_{A=a} \mid L)}{f(a \mid L)}\right] = E_L E\left(\frac{y 1_{A=a}}{f(a \mid L)} \,\Big|\, L\right) = E\left[\frac{y 1_{A=a}}{f(a \mid L)}\right]. \; \square$$

EFFECT MODIFICATION (CHAP 4)

Interest is in $E(Y^a \mid M=m)$, some M

Causal effect in the treated if $E(Y^0 \mid A=1) \neq E(Y^1 \mid A=1)$

THM Under (C) + (CE) + (P)

$$E(Y^a \mid A=a') = \frac{E\left[Y 1_{A=a} \frac{f(a' \mid L)}{f(a \mid L)}\right]}{E\left[1_{A=a} \frac{f(a' \mid L)}{f(a \mid L)}\right]}$$

<u>proof</u> $E(Y^a 1_{A=a'} \mid L) = E(Y^a \mid L, A=a') f(a' \mid L) \overset{(CE)}{=} E(Y^a \mid L, A=a) f(a' \mid L)$

$$\overset{(C)}{=} E(Y^{\cancel{a}} \mid L, A=a) f(a' \mid L) = E(Y 1_{A=a} \mid L) \frac{f(a' \mid L)}{f(a \mid L)}$$

So $E\left[Y^a 1_{A=a'}\right] = E\left[Y 1_{A=a} \frac{f(a' \mid L)}{f(a \mid L)}\right]$. Similarly $E 1_{A=a'} = E\left[1_{A=a} \frac{f(a' \mid L)}{f(a \mid L)}\right]$

INTERACTION OF TREATMENTS (CHAP 5)

$A, E \in \{0,1\}$

Counterfactuals $Y^{0,0}, Y^{0,1}, Y^{1,0}, Y^{1,1}$.

Additive interaction $P(Y^{1,1}=1) - P(Y^{0,0}=1) - \left[P(Y^{1,0}=1) - P(Y^{0,0}=1) + P(Y^{0,1}=1) - P(Y^{0,0}=1)\right]$.

Multiplicative interaction

$$\frac{P(Y^{1,1}=1)}{P(Y^{0,0}=1)} - \frac{P(Y^{1,0}=1)}{P(Y^{0,0}=1)} \frac{P(Y^{0,1}=1)}{P(Y^{0,0}=1)}.$$

INTERMEZZO: PROBABILISTIC DAGs

Directed graph: collection V of nodes with arrows

DAG: directed graph without directed cycles.

A random vector $X = (X_v : v \in V)$ factorizes over a DAG if its density p satisfies

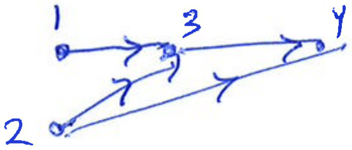
$$\text{(F)} \qquad p(x) = \prod_{v \in V} p(x_v \mid x_{pa(v)})$$

where $pa(v)$: parents of v in DAG

$x_W = (x_v : v \in W)$

$x_v \mapsto p(x_v \mid x_{pa(v)})$ is density of $X_v \mid X_{pa(v)}$

EX



$$p(x_1, x_2, x_3, x_4) = p(x_1)p(x_2)p(x_3 \mid x_1, x_2)\,p(x_4 \mid x_2, x_3)$$

THM (F) is equivalent to (see Lauritzen-book)

$X_A \perp\!\!\!\perp X_B \mid X_S$ for all disjoint $A, B, S \subset V$ such that S d-separates A and B.

DEF A and B are d-separated by S if every path from $a \in A$ to $b \in B$ either passes through a noncollider $s \in S$ or a collider $t \notin S$ none of whose descendants are in S.

collider

noncollider

Such a path is blocked or closed (by S). Otherwise open.

S are the variables we condition on

A path is closed relative to conditioning on X_S.

$S = \emptyset$ corresponds to unconditional (in)dependence

- $X_A \perp\!\!\!\perp X_B$ if every path from A to B contains a collider
- if a path from A to B is open relative to $S = \emptyset$, then it consists of noncolliders.

  Conditioning on any element of the path closes it

$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend left] & A \arrow[r] & Y \end{tikzcd}$$

path $A \to Y$ is open $\Rightarrow$ not $A \perp\!\!\!\perp Y$
path $A \leftarrow L \to Y$ is open. closed by conditioning on L

$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend left] & A & Y \end{tikzcd}$$

not $A \perp\!\!\!\perp Y$, but $A \perp\!\!\!\perp Y \mid L$

CAUSAL DAGS AND CONFOUNDING (CHAP 6-7)

A DAG is a <u>causal DAG</u> if

- an arrow indicates a direct effect and absence of an arrow nonexistence of a direct effect
- all common causes of 2 or more variables (also if unmeasured) are represented by nodes (but . .)
- any variable is understood to be cause of its descendants.

Not a math. definition but based on expert knowledge.

EXAMPLE 2

$$\begin{tikzcd} & L \arrow[r] & A \arrow[r] & Y \\ U \arrow[ur] \arrow[urrr] & & & \end{tikzcd}$$

U: atherosclerosis
L: heart disease
A: asperine
Y: stroke

"confounding by indication"

EXAMPLE 3

$$\begin{tikzcd} & L \arrow[dr] & \\ U \arrow[ur] \arrow[r] & A \arrow[r] & Y \end{tikzcd}$$

U: personality, social factors
L: smoking
A: exercise
Y: death

"reserve causation"

EXAMPE 4 | OR

U: population stratification
A: genetic marker
L: another marker
Y: trait

EXAMPLE 1

$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend left] & A \arrow[r] & Y \end{tikzcd}$$

L: physical condition
A: work as firefighter
Y: death
"healthy worker bias".

CAUSAL DAGs and (CE).

Exchangeability (ie. $Y^a \perp\!\!\!\perp A \mid L$) is translated into graph language as the lack of open paths between the treatment A and outcome Y nodes — other than those originating from A — that would result in an association between A and Y. (pg1).

Ideas

- Interpret causal graph as probabilistic DAG
- Consider subgraph with arrows outgoing from A removed.

If in this subgraph
- $Y \perp\!\!\!\perp A$, then all association between Y and A is causal
- $Y \perp\!\!\!\perp A \mid L$, then all conditional association " " "

Back door path is an open path between A and Y in the graph with arrows outgoing from A removed

If all back door paths are closed (possibly after conditioning on L), then we are in situation of the bullets.

Confounding (of effect of A on Y) is existence of open backdoor path

Confounder is a nondescendant of A that can be used to close a backdoor path.

(differs from 'traditional' definition ! L is confounder if
- L and A associated
- L and Y associated given A
- L not on causal path $A \to Y$

EXAMPLE 1

$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend left] & A \arrow[r] & Y \end{tikzcd}$$

reduced

$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend left] & A & Y \end{tikzcd}$$

$A \leftarrow L \rightarrow Y$ is open, hence backdoor

$A \leftarrow \boxed{L} \rightarrow Y$ is closed by conditioning on L

L confounder.

EXAMPLE 2

$$\begin{tikzcd} L \arrow[r] & A \arrow[r] & Y \\ U \arrow[u] \arrow[urr] & & \end{tikzcd}$$

reduced

$$\begin{tikzcd} L \arrow[r] & A & Y \\ U \arrow[u] \arrow[urr] & & \end{tikzcd}$$

$A \leftarrow L \leftarrow U \rightarrow Y$ is open, backdoor can be closed by conditiong on L

EXAMPLE 3

$$\begin{tikzcd} U_1 \arrow[d] \arrow[drr] & & \\ L & A \arrow[r] & Y \\ U_2 \arrow[u] \arrow[ur] & & \end{tikzcd}$$

U_1 = precancer lesion
U_2 = health conscious
L = pap smear
A = physical activity
Y = cervical cancer

reduced

$$\begin{tikzcd} U_1 \arrow[d] \arrow[drr] & & \\ L & A & Y \\ U_2 \arrow[u] \arrow[ur] & & \end{tikzcd}$$

$A \leftarrow U_2 \rightarrow L \leftarrow U_1 \rightarrow Y$ is backdoor path, but closed (at collider L) no conditioning necessary

L not confounder

CAUSAL DAGS and COUNTERFACTUALS

Interpret DAG as NSEM:

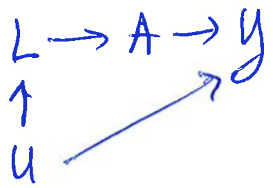
LEMMA Let $V=\{1,\dots,m\}$ with nodes ordered so that parents precede children. Then there exist measurable functions and iid. uniform variables $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$ such that

$$X_1 = f_1(\varepsilon_1)$$
$$X_2 = f_2(X_{pa(2)}, \varepsilon_2)$$
$$\vdots$$
$$X_j = f_j(X_{pa(j)}, \varepsilon_j)$$
$$\vdots$$
$$X_m = f_m(X_{pa(m)}, \varepsilon_m).$$

proof If X's real-valued, can use (conditional) quantile functions. More general, this works for X's with values in Polish space by Borel isomorphy. □

Form counterfactuals for intervention on X_j by replacing every occurrence of X_j on right side by x_j.

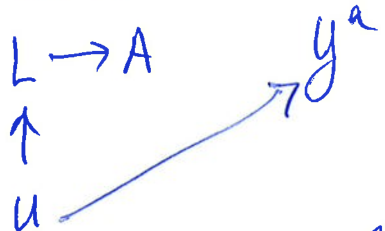
EXAMPLE



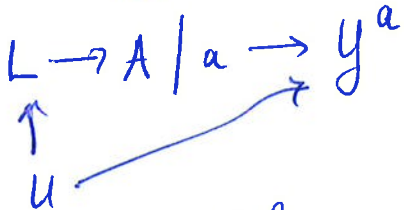
$$U = f_1(\varepsilon_1)$$
$$L = f_2(U, \varepsilon_2)$$
$$A = f_3(L, \varepsilon_3)$$
$$Y = f_4(U, A, \varepsilon_4)$$

give counterfactuals for intervention on A

$$U = f_1(\varepsilon_1)$$
$$L = f_2(U, \varepsilon_2)$$
$$A = f_3(L, \varepsilon_3)$$
$$Y^a = f_4(U, a, \varepsilon_4)$$



or SWIG



Note: $Y^a \perp\!\!\!\perp A \mid L$ because path $A \leftarrow L \leftarrow U \rightarrow Y^a$ is closed by conditioning on L

THM If there is no backdoor path from A to Y after conditioning on the nondescendant L of A, then $Y^a \perp\!\!\!\perp A \mid L, \forall a$.

Consequently, $Y^a \mid L \sim Y \mid L, A=a$.

<u>proof</u> The set of counterfactual variables possesses conditional distributions corresponding to the DAG G obtained by deleting outgoing arrows from A from the original DAG. All paths between Y^a and A in this DAG are back door path, by definition. If these are closed after conditioning on L, then $Y^a \perp\!\!\!\perp A \mid L$, by the general theory on probabilistic graphical models $\square$.

Note: Hernan & Robins point out that $\varepsilon_1 \cdots \varepsilon_m$ should not always be taken independent.

Question: Is the counterfactual setup as obtained here "unique" in an appropriate sense?

CAN WE DERIVE A CAUSAL DAG FROM DATA?

No! For instance with two variables A, Y, there are 3 possible causal DAGs

(1)

(2)

(3)

From the distribution of (A, Y) you can choose between (1) and ((2) or (3)) (because independence vs dependence), but not between (2) and (3).